One-rooted trees attached to a triangular lattice surface

LETTER TO THE EDITOR

One-rooted trees attached to a triangular lattice surface

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Abstract. The number of distinct weakly embedded n vertex one-rooted trees attached to the surface of the triangular lattice at the root have been enumerated for trees with $n \leq 12$. Writing the generating function $G(p) = \sum_n n C_n p^n$, $C_n \sim_{n \to \infty} \lambda^n n^{-\theta_1}$ indicates $\theta_1 = 2.008 \pm 0.010$.

The association of branched chemical isomers with the study of tree graphs dates back (at least) to the work of Cayley (1874) and the growth of branched polymers is often quoted as a possible realisation of the lattice tree problem (e.g. Lubensky and Isaacson 1979). Our own interest in the problem of trees attached to a surface was stimulated by the description of work on viscous fingers by Nittmann *et al* (1985) in which a low-viscosity fluid is injected into a high-viscosity fluid contained in a thin cell. The low-viscosity fluid is injected at a point (the 'root') on the edge of the cell and was found to spread into the high-viscosity fluid by forming a branched structure with a fractal dimension of $d_f \simeq 1.39$.

With the above physical systems in mind, we have enumerated the number of distinct weak embeddings for n vertex one-rooted trees on the triangular lattice when the root is embedded in the surface of the lattice and $n \leq 12$.

The analogous problem for trees embedded in the bulk of the lattice and $n \leq 11$ has previously been analysed by Whittington *et al* (1983). For trees embedded in the bulk the generating function for the number of embeddings $n c_n$ is

$$G(p) = \sum_{n=1} n c_n p^n \tag{1}$$

where

$$c_n \underset{n \to \infty}{\sim} n^{-\theta} \lambda^n. \tag{2}$$

By analogy, we write the moments of the generating function for trees attached to a surface as

$$G_1^{(m)}(p) = \sum_{n=0} n^{m+1} C_n p^n \tag{3}$$

where $n C_n$ is the number of embeddings for n vertex trees rooted at the surface and we assume

$$C_n \underset{n \to \infty}{\sim} n^{-\theta_1} \lambda^n. \tag{4}$$

The values of $n C_n$ for $n \leq 12$ are given in table 1.

Table 1. Total number of weak embeddings (nC_n) for one-rooted trees attached to the triangular lattice surface.

n	nC_n
1	1
2	4
3	22
4	138
5	927
6	6480
7	46 633
8	342 688
9	2558 716
10	19 347 190
11	147 785 296
12	1138 384 800

From (4) we expect the logarithmic derivative of $G_1^{(0)}$ to have a simple pole at $p = p_c = 1/\lambda$ with residue $-2 + \theta_1$. Padé approximants to this series appeared to be converged around 0.1207 ± 0.0009 with $-2 + \theta_1 \simeq 0.30$. This value of p_c is too high to be consistent with the value of $p_c = 0.1189 \pm 0.0003$ obtained by Whittington *et al* (1983) from the bulk series. However the Padé approximant method is often unreliable when the original series ($G_1^{(0)}$ in this case) has a weak singularity. We therefore analysed the series for $G_1^{(1)}$ by the same method. In this case, the latter approximants considered all gave $p_c = 0.1189$.

We then used the Baker–Hunter (1973) method to check for affects due to corrections to scaling. (The application of this method has previously been described for self-avoiding walks attached to a surface (De'Bell and Lookman 1985) and in that case, gave excellent agreement with the value of the critical point obtained by Guttmann (1984) from the bulk self-avoiding walk series.)

The scatter in the estimates of $3 - \theta_1$ obtained from the Baker–Hunter method for $G_1^{(1)}$ for various trial values of p_c is illustrated in figure 1. We find (as with the self-avoiding walk case) that there is a relatively narrow region of trial values of p_c for which the approximants are well converged and that outside this region the convergence becomes rapidly worse. From this region of best convergence we obtain

$$p_c = 1/\lambda = 0.118\,90 \pm 0.000\,05 \tag{5}$$

(in excellent agreement with Whittington *et al*'s result (1983)), and from the same region
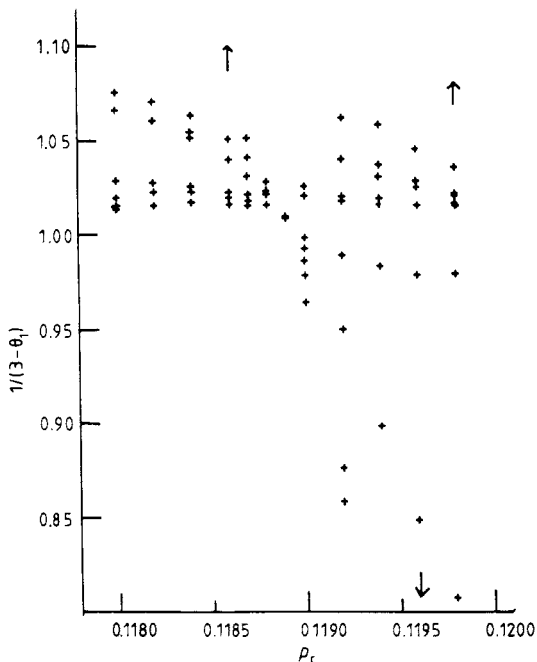
$$\theta_1 = 2.008 \pm 0.010 \tag{6}$$

where the error bounds on θ_1 indicate only the variation in the Padé approximant poles over the range of p_c given in (5).

From (3)

$$G_1^{(1)}(p) \underset{p \to p_c^-}{=} A_1(1 - p/p_c)^{-3 + \theta_1} + \text{less singular terms} \tag{7}$$

and the residues of the Padé approximants to the auxiliary function formed in the Baker–Hunter method are estimates to $-A_1/(\theta_1 - 3)$; hence we obtain

$$A_1 = 0.937 \pm 0.001, \tag{8}$$

Figure 1. Poles (+) of the [4/4], [5/4], [6/4], [4/5], [5/5], [6/5] and [5/6] Padé approximants to the Baker-Hunter auxiliary function (see text) for surface trees on the triangular lattice. Arrows indicate that a pole occurs outside the range of the diagram.

where the error bounds are determined only from the variation in the residues of the approximants at $p_c = 0.1189$.

As a check on the consistency of our method we repeated the analysis for the first moment of the bulk series (using the 11 terms listed by Whittington *et al* 1983)

$$G^{(1)}(p) = A(1 - p/p_c)^{-3+\theta} + \text{less singular terms.} \tag{9}$$

(In this case the Padé approximants to the auxiliary function to $G^{(0)}(p)$ exhibit interfering defects which prevent convergence.) A plot similar to figure 1 is obtained with a region of best convergence centred on $p_c = 0.1189$ and from this plot

$$\theta = 1.01 \pm 0.03, \tag{10}$$

in excellent agreement with previous estimates (Whittington *et al* 1983). From the residues of the approximants to the auxiliary function

$$A = 0.50 \pm 0.02. \tag{11}$$

(The error bounds in (10) and (11) have the same meaning as those in (5) and (6) respectively.)

In summary, a method of series analysis due to Baker and Hunter (1973) has been applied to the first moment of the generating function for weak embeddings of one-rooted trees on a triangular lattice with the root at the surface of the lattice. This leads to the value of θ_1 given in the abstract and value of $p_c = 1/\lambda$ in excellent agreement with previous estimates based on the bulk series.

Finally we note that field theory (Lubensky and Isaacson 1979) and renormalisation group analysis (Family 1980) indicates that the prefactor exponent θ is the same for trees and lattice animals in the bulk. In view of our present work an extension of these results to surface trees and animals would be of considerable interest.

References

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